

TWO RESULTS ON ENTROPY, CHAOS, AND INDEPENDENCE IN SYMBOLIC DYNAMICS

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ABSTRACT. We survey the connections between entropy, chaos, and independence in topological dynamics. We present extensions of two classical results placing the following notions in the context of symbolic dynamics:

- (1) Equivalence of positive entropy and the existence of a large (in terms of asymptotic and Shnirelman densities) set of combinatorial independence for shift spaces.
- (2) Existence of a mixing shift space with a dense set of periodic points with topological entropy zero and without ergodic measure with full support, nor any distributionally chaotic pair.

Our proofs are new and yield conclusions stronger than what was known before.

1. INTRODUCTION.

Furstenberg [12, p. 38] calls a dynamical system *deterministic* if its *topological entropy* vanishes. One may argue that the future of a deterministic dynamical system can be predicted if its past is known (see [40, Chapter 7]). In a similar way *positive entropy* may be related to *randomness* and *chaos*.

This article surveys the relations between entropy, independence, and chaos, adding a new twist to two classical results.

The first result describes in what sense positive entropy may be understood as independence. It turns out that for a dynamical system there is a deep connection between positive entropy and randomness defined through the notion of the *combinatorial independence*. General results of this kind can be found in [20]. A special instance of this result (see [40, Theorem 8.1]) says that a symbolic dynamical system (a shift space) $X \subset \{0, 1\}^{\mathbb{N}}$ has positive entropy if and only if there is a large set of indices along which points from X behave like $\{0, 1\}$ -valued independent random variables. We strengthen this theorem by proving that positive topological entropy of a shift space is equivalent to the existence of an independence set for which asymptotic **and** Shnirelman densities are equal and positive. This shows that, for a shift space with positive entropy, one can find an independence set which is large and structured. One can easily adapt our reasoning to the general context.

Before we describe our second result, let us consider the following properties that a dynamical system (X, T) may have.

- (1) There exists an ergodic T -invariant Borel probability measure on X with full support (that is, an invariant measure positive on all nonempty open subsets of X).

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- (2) The system (X, T) is *topologically transitive*, that is, for any pair of nonempty open sets $U, V \subset X$ there exists a positive integer n with $T^{-n}(U) \cap V \neq \emptyset$.
- (3) There exists a T -invariant (not necessarily ergodic) Borel probability measure on X with full support.

It is straightforward that (1) implies (2) and, trivially, (3). It is also relatively easy to prove that (2) does not imply (3) and hence neither implies (1). According to [39], H. Keynes asked whether (2) and (3) together imply (1). B. Weiss answered this question negatively in [39]. He constructed a transitive shift space with a dense set of periodic points and no ergodic invariant measure of full support. He also noted, omitting the proof, that his example is in fact topologically mixing and has zero topological entropy. Furthermore, Weiss conjectured [39, Remark 1] that the only ergodic measures for his system are those concentrated on the orbit of a single periodic point. Note that Huang and Ye constructed a uniformly positive entropy system without an ergodic invariant measure of full support (see [17, Theorem 9.6]).

We provide an alternative construction of a similar example. Our method allows us to prove that the system we have defined possesses all the properties proved or conjectured about the example from [39]. Furthermore, we show that our example does not possess any distributionally chaotic pairs. This extends a result by Oprocha [28] who sketched a construction of a Devaney chaotic system without any DC1 distributionally chaotic pairs. In addition, we show that our example is ω -chaotic, but not ω^* -chaotic. To do this we provide a new method of constructing ω -chaotic sets.

2. NOTATION AND DEFINITIONS.

Let \mathbb{N} denote the set of *positive* integers. We denote the number of elements of a finite set A by $|A|$. Given any real number x , we write $\lfloor x \rfloor$ for the largest integer not greater than x . A sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is *subadditive* if $a_{m+n} \leq a_m + a_n$ for all $m, n \in \mathbb{N}$. We recall that Fekete's Lemma states that if a sequence $\{a_n\}_{n=1}^{\infty}$ of nonnegative numbers is subadditive, then the sequence $\{\frac{1}{n}a_n\}_{n=1}^{\infty}$ converges to a limit equal to the infimum of the terms of this sequence.

A (*topological*) *dynamical system* is a pair (X, T) , where X is a compact metric space and $T: X \rightarrow X$ is a continuous map. By ρ we denote a compatible metric for X . A dynamical system (X, T) is *transitive* if for any nonempty open sets $U, V \subset X$ there is $n \in \mathbb{N}$ such that $T^{-n}(V) \cap U \neq \emptyset$. We say that (X, T) is *topologically mixing* if for any nonempty open sets $U, V \subset X$ there is $N \in \mathbb{N}$ such that $T^{-n}(V) \cap U \neq \emptyset$ for all $n \geq N$. The set of limit points of the sequence $\{T^n(x)\}_{n \in \mathbb{N}}$ is called the ω -*limit set* of x and denoted by $\omega_T(x)$. A set $M \subset X$ is called *minimal* if it is nonempty, closed, invariant, and contains no proper subset with these properties.

We assume the reader is familiar with the basic notions of topological dynamics and ergodic theory (see [38]). In particular, we assume that the reader knows the definition of topological entropy.

3. DEFINITIONS OF CHAOS AND THEIR RELATIONS WITH TOPOLOGICAL ENTROPY

In this section we collect some definitions of chaos and survey connections between them and topological entropy.

3.1. Li-Yorke chaos. Let (X, T) be a dynamical system and $(x, y) \in X \times X$. We say that (x, y) is a *Li-Yorke pair*¹ if

$$\begin{aligned} \liminf_{n \rightarrow \infty} \rho(T^n(x), T^n(y)) &= 0, \\ \limsup_{n \rightarrow \infty} \rho(T^n(x), T^n(y)) &> 0. \end{aligned}$$

A dynamical system (X, T) is *Li-Yorke chaotic* if there is an uncountable set $S \subset X$ such that every pair (x, y) with $x, y \in S$ and $x \neq y$ is a Li-Yorke pair. A nice survey of properties of Li-Yorke chaotic systems can be found in [8].

3.2. Distributional chaos. Given $x, y \in X$ and $t \in \mathbb{R}$ we define an *upper* and *lower distribution* function by

$$\begin{aligned} F_{xy}(t) &= \liminf_{n \rightarrow \infty} \frac{1}{n} |\{0 \leq j \leq n-1 : \rho(T^j(x), T^j(y)) < t\}|, \\ F_{xy}^*(t) &= \limsup_{n \rightarrow \infty} \frac{1}{n} |\{0 \leq j \leq n-1 : \rho(T^j(x), T^j(y)) < t\}|. \end{aligned}$$

Clearly, $t \mapsto F_{xy}(t)$ and $t \mapsto F_{xy}^*(t)$ are nondecreasing, $0 \leq F_{xy}(t) \leq F_{xy}^*(t) \leq 1$ for all $t \in \mathbb{R}$, $F_{xy}(t) = F_{xy}^*(t) = 0$ for all $t \leq 0$, and $F_{xy}(t) = F_{xy}^*(t) = 1$ for all $t > \text{diam } X$. Distributional chaos was introduced in [35]. Following [3] we say that a pair $(x, y) \in X \times X$ is a *DC1-pair* if $F_{xy}^*(t) = 1$ for all $t > 0$, and $F_{xy}(s) = 0$ for some $s > 0$. A pair (x, y) is a *DC2-pair* if $F_{xy}^*(t) = 1$ for all $t > 0$, and $F_{xy}(s) < 1$ for some $s > 0$. Finally, a *DC3-pair* is a pair $(x, y) \in X \times X$ such that $F_{xy}(t) < F_{xy}^*(t)$ for all t in some interval of positive length.

Let $i \in \{1, 2, 3\}$. The dynamical system (X, T) is *distributionally chaotic of type i* (or *DC i -chaotic* for short), if there is an uncountable set $S \subset X$ such that any pair of distinct points from S is a DC i -pair.

3.3. Devaney chaos. Devaney calls a dynamical system (X, T) *chaotic* if T is transitive, T -periodic points are dense in X , and T has sensitive dependence on initial conditions (see [9]). It turns out that if X is infinite, then the sensitive dependence follows from the other two conditions (see [4, 14]). Since we restrict our attention to compact metric spaces without isolated points we say that (X, T) is Devaney chaotic if T is transitive and T -periodic points are dense in X . Such systems are also known as *P-systems* (see [14]).

3.4. ω -chaos. We say that a dynamical system (X, T) is *ω -chaotic* if there exists an uncountable set $S \subset X$ such that for any $x, y \in S$ with $x \neq y$ we have

- $\omega_T(x) \setminus \omega_T(y)$ is uncountable,
- $\omega_T(x) \cap \omega_T(y)$ is nonempty,
- $\omega_T(x)$ is not contained in the set of periodic points.

Li [24] introduced ω -chaos and proved that it is equivalent to positive topological entropy for interval maps. In [25] Li defined a variant of ω -chaos, called ω^* -chaos. A dynamical system (X, T) is *ω^* -chaotic* if there exists an uncountable set $S \subset X$ such that for any $x, y \in S$ with $x \neq y$ we have

- $\omega_T(x) \setminus \omega_T(y)$ contains an infinite minimal set,
- $\omega_T(x) \cap \omega_T(y)$ is nonempty.

¹Such pairs appeared in the seminal paper of Li and Yorke [26].

Since an infinite minimal set is uncountable and does not contain a periodic point, ω^* implies ω -chaos. All examples of ω -chaotic maps known to us are in fact ω^* -chaotic. As is common for various notions of chaos in a general setting of an arbitrary compact metric space, ω -chaos is independent of most other notions of chaos.

3.5. Entropy vs. chaos. There is no connection between positive topological entropy and Devaney chaos. It is known that Devaney chaos implies Li-Yorke chaos [16] and does not imply distributional chaos [28]. The example in [39] shows that Devaney chaos (hence Li-Yorke chaos as well by [16]) does not imply positive entropy. In general, positive entropy neither implies topological transitivity nor existence of periodic points. Nevertheless, Li [24] proved that for maps on the interval positive topological entropy is equivalent to the existence of a subsystem chaotic in the sense of Devaney. The question whether positive topological entropy implies Li-Yorke chaos or DC2 distributional chaos remained open for some time, but eventually both implications turned out to be true; see [6] and [10], respectively. The consequences of positive topological entropy for dynamics of pairs and tuples were also examined in [7, 18]. Distributional chaos of type 2 implies Li-Yorke chaos by definition. The converse implication is not true, because if $X = [0, 1]$, then by [35] distributional chaos DC3 is equivalent to the positive topological entropy, and there are Li-Yorke chaotic interval maps with zero topological entropy (as was shown independently by Smítal [37] and Xiong [41]). Pikula proved that there is no connection between positive topological entropy and DC1 distributional chaos [32], therefore Li-Yorke chaos doesn't imply positive topological entropy, either. Pikula [32] also constructed an example of ω -chaotic dynamical system without Li-Yorke pairs. Since there exist minimal systems with positive topological entropy, none of the following properties: positive topological entropy, distributional chaos, Li-Yorke chaos imply ω -chaos (just note that if (X, T) is minimal, then $\omega_T(x) = X$ for every $x \in X$). Downarowicz and Ye [11] constructed a transitive dynamical system (X, T) which is Devaney chaotic and every point $x \in X$ is either transitive ($\omega_T(x) = X$) or periodic. Such a system cannot be ω -chaotic.

All above is only a glimpse of the vast literature of the subject. For deeper discussion of these matters we refer the reader to the excellent surveys by Blanchard [5], Glasner and Ye [13], Li and Ye [23].

4. SYMBOLIC DYNAMICS

With regard to symbolic dynamics we follow the notation and terminology of Lind and Marcus [27] as closely as possible, except that we consider only one-sided shifts.

Definitions (Full shifts). Let \mathcal{A} be a nonempty finite set. We call \mathcal{A} the *alphabet* and elements of \mathcal{A} are *symbols*. The *full \mathcal{A} -shift* is the set

$$\mathcal{A}^{\mathbb{N}} = \{x = \{x_i\}_{i=1}^{\infty} : x_i \in \mathcal{A} \text{ for all } i \in \mathbb{N}\}.$$

We equip \mathcal{A} with the discrete topology and $\mathcal{A}^{\mathbb{N}}$ with the product topology. We usually write an element of $\mathcal{A}^{\mathbb{N}}$ as $x = \{x_i\}_{i=1}^{\infty} = x_1x_2x_3\dots$. Then $\mathcal{A}^{\mathbb{N}}$ is a compact topological space and the formula

$$(1) \quad \rho(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 2^{-\min\{j \in \mathbb{N} : x_j \neq y_j\}}, & \text{if } x \neq y, \end{cases}$$

defines a metric on $\mathcal{A}^{\mathbb{N}}$ which is compatible with the product topology. The *shift map* $\sigma: \mathcal{A}^{\mathbb{N}} \rightarrow \mathcal{A}^{\mathbb{N}}$ is a continuous transformation given by

$$x = \{x_i\}_{i=1}^{\infty} \mapsto \sigma(x) = \{x_{i+1}\}_{i=1}^{\infty}.$$

That is, $\sigma(x)$ is the sequence obtained by dropping the first symbol of x . A *full r -shift* is the full shift over the alphabet $\{0, 1, \dots, r-1\}$ and a *full binary shift* is the full 2-shift.

Definitions (Blocks). A *block over \mathcal{A}* is a finite sequence of symbols and its *length* is the number of its symbols. An *n -block* stands for a block of length n . The *empty block*, denoted by ε , is the unique block with no symbols and length zero. The set of all blocks over \mathcal{A} (including ε) is denoted by \mathcal{A}^* . The *concatenation* of two blocks $u = a_1 \dots a_k$ and $v = b_1 \dots b_l$ is the block $uv = a_1 \dots a_k b_1 \dots b_l$. We write u^n for the concatenation of $n \geq 1$ copies of a block u and u^{∞} for the sequence $uuu \dots \in \mathcal{A}^{\mathbb{N}}$.

By $x_{[i,j]}$ we denote the block $x_i x_{i+1} \dots x_j$, where $1 \leq i \leq j$ and $x = (x_i)_{i=1}^{\infty} \in \mathcal{A}^{\mathbb{N}}$. We say that a block $w \in \mathcal{A}^*$ *occurs in x* and x *contains w* if $w = x_{[i,j]}$ for some integers $1 \leq i \leq j$. Note that ε occurs in every sequence from $\mathcal{A}^{\mathbb{N}}$. Similarly, given an n -block $w = w_1 \dots w_n \in \mathcal{A}^*$ we define $w_{[i,j]} = w_i w_{i+1} \dots w_j \in \mathcal{A}^*$ for each $1 \leq i \leq j \leq n$. A *prefix* of a block $z \in \mathcal{A}^*$ is any u such that $z = uv$ for some $v \in \mathcal{A}^*$.

Definitions (Shift spaces). Given any collection \mathcal{F} of blocks over \mathcal{A} (i.e., a subset of \mathcal{A}^*) we define a *shift space specified by \mathcal{F}* , denoted by $X_{\mathcal{F}}$, as the set of all sequences from $\mathcal{A}^{\mathbb{N}}$ which do not contain any blocks from \mathcal{F} . We say that \mathcal{F} is a collection of *forbidden blocks for $X_{\mathcal{F}}$* .

A *shift space* is a set $X \subset \mathcal{A}^{\mathbb{N}}$ such that $X = X_{\mathcal{F}}$ for some $\mathcal{F} \subset \mathcal{A}^*$. A *binary shift space* is a shift space over the alphabet $\{0, 1\}$. Equivalently, $X \subset \mathcal{A}^{\mathbb{N}}$ is a shift space if it is a closed and σ -invariant subset of $\mathcal{A}^{\mathbb{N}}$.

Definition (Language of a shift space). The *language* of a shift space X over \mathcal{A} is the set of blocks over \mathcal{A} which do occur in some sequence $x \in X$. We denote it by $\mathcal{B}(X)$. We write $\mathcal{B}_n(X)$ for the set of all n -blocks contained in $\mathcal{B}(X)$. Similarly, $\mathcal{B}(x) \subset \mathcal{A}^*$ is the collection of all blocks occurring in $x \in \mathcal{A}^{\mathbb{N}}$. The language of a shift space determines the shift space: $x \in \mathcal{A}^{\mathbb{N}}$ belongs to a shift space X if and only if for every $k \in \mathbb{N}$ the initial block $x_{[1,k]}$ is in $\mathcal{B}(X)$.

A *cylinder* of an n -word $w \in \mathcal{A}^*$ in a shift space X is the set

$$[w]_X = \{x \in X : x_{[1,n]} = w\}.$$

If a space X is clear from the context, we call $[w]_X$ the cylinder of w . The collection of all cylinders $\{[w]_X : w \in \mathcal{B}(X)\}$ is a basis of the topology of X . If \mathcal{L} is a language of some shift space over \mathcal{A} , then \mathcal{L} is *factorial*, meaning that if $u \in \mathcal{L}$ and $u = vw$ for some blocks $v, w \in \mathcal{A}^*$, then both v and w also belong to \mathcal{L} , and *prolongable*, meaning that for every block u in \mathcal{L} there is a symbol $a \in \mathcal{A}$ such that ua also belongs to \mathcal{L} .

Conversely, every factorial and prolongable subset $\mathcal{L} \subset \mathcal{A}^*$ determines a shift space $X_{\mathcal{L}}$ such that \mathcal{L} is the language of X (see [27, Proposition 1.3.4]). A collection of forbidden blocks defining X is $\mathcal{F} = \mathcal{A}^* \setminus \mathcal{L}$. A point $x \in \mathcal{A}^{\mathbb{N}}$ is in $X_{\mathcal{L}}$ if and only if $x_{[i,j]} \in \mathcal{L}$ for all $i, j \in \mathbb{N}$ with $i < j$.

We will use superscripts in brackets to denote indices for sequences of blocks in \mathcal{A}^* or points in $\mathcal{A}^{\mathbb{N}}$. That is, we write $\{w^{(n)}\}_{n=1}^{\infty}$ for a sequence of blocks and we use subscripts for enumerating symbols in the block: $w^{(n)} = w_1^{(n)} w_2^{(n)} \dots w_k^{(n)}$.

In its full generality, the concept of entropy was defined by Adler, Konheim and McAndrew [1] for any continuous map $T: X \rightarrow X$ on an arbitrary compact topological space X . The definition below, specific to symbolic dynamics, is equivalent to the general one.

Definition (Entropy). Let $X \subset \mathcal{A}^{\mathbb{N}}$ be a nonempty shift space and let $m, n \in \mathbb{N}$. Observe that $|\mathcal{B}_{m+n}(X)| \leq |\mathcal{B}_m(X)| \cdot |\mathcal{B}_n(X)|$, and hence

$$\log |\mathcal{B}_{m+n}(X)| \leq \log |\mathcal{B}_m(X)| + \log |\mathcal{B}_n(X)|.$$

Using Fekete's lemma we define the *entropy of X* , denoted by $h(X)$, as

$$h(X) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |\mathcal{B}_n(X)| = \inf_{n \geq 1} \frac{1}{n} \log |\mathcal{B}_n(X)|.$$

5. ENTROPY AND INDEPENDENCE IN SYMBOLIC DYNAMICS

Independence is certainly among the most popular terms used to single out an interesting mathematical phenomenon. Consider the full shift over $\{0, 1\}$. Clearly, the full shift is as random as possible — it is a model containing all possible outcomes of an infinite sequence of fair coin tosses (we set $x_i = 1$ if the coin turns heads on the i -th toss, and $x_i = 0$ otherwise). Anything that can happen is encoded in some point x — more precisely, for any $J \subset \mathbb{N}$ and any assignment $\varphi: J \rightarrow \mathcal{A}$ of 0's and 1's to elements of J there is a point x in the full shift realizing this assignment, that is, $x_i = \varphi(i)$ for all $i \in J$. For a general shift space X over $\{0, 1\}$ the above procedure may not work for every set $J \subset \mathbb{N}$. But one may still consider the following problem: let X be a shift space over \mathcal{A} . Assume that someone picks a set $J \subset \mathbb{N}$ and for each $i \in J$ chooses a symbol $\varphi(i) \in \mathcal{A}$ (the choice may be random). Does there exist a point $x \in X$ whose i -th coordinate is $\varphi(i)$ for every $i \in J$? If the answer is positive for every assignment φ , then we say that X is *independent over J* . Now we may “measure” the randomness of X by the size of the “largest” independent set $J \subset \mathbb{N}$.

Definition (Independence set for a shift space). We say that a set $J \subset \mathbb{N}$ is an *independence set* for a shift space $X \subset \mathcal{A}^{\mathbb{N}}$ if for every function $\varphi: J \rightarrow \mathcal{A}$ there is a point $x = \{x_j\}_{j=1}^{\infty} \in X$ such that $x_i = \varphi(i)$ for every $i \in J$.

Note that if $J \subset \mathbb{N}$ is an independence set for a shift space X , then so is every subset of J .

5.1. Asymptotic density and Shnirelman density. The most natural way to describe the size of an infinite subset of \mathbb{N} is the *asymptotic density*. It is translation invariant and it is invariant under the exclusion or inclusion of finitely many elements. A similar notion is the *Shnirelman density*, which formally is also a limit, but it gives information about the structure of $A \cap \{1, \dots, n\}$ for every $n \in \mathbb{N}$. Although it seems to be less natural a concept than asymptotic density, Shnirelman's notion proved invaluable to his approach to the Goldbach problem.

Definitions (Densities). A set $A \subset \mathbb{N}$ has *asymptotic density* α if the limit

$$d(A) = \lim_{n \rightarrow \infty} \frac{|A \cap \{1, 2, \dots, n\}|}{n}$$

exists and is equal to α .

The *Shnirelman density* $d_{\text{Sh}}(A)$ of a set $A \subset \mathbb{N}$ is defined by

$$d_{\text{Sh}}(A) = \inf \left\{ \frac{|A \cap \{1, 2, \dots, n\}|}{n} : n \in \mathbb{N} \right\}.$$

5.2. Positive entropy is equivalent to independence. Both notions measure how dense a set is, and a set with any of these densities positive may be called large. We will prove that positive topological entropy is equivalent to existence of an independence set for which asymptotic **and** Shnirelman densities are equal and positive. This shows that for a shift space with positive entropy one can find an independence set which is large and structured (there exists $0 < \alpha \leq 1$ such that our set occupies at least α proportion of $\{1, \dots, n\}$ for *every* $n \in \mathbb{N}$). Note that positive asymptotic density of a set A tells us only that $|A \cap \{1, \dots, n\}|$ behaves like αn for some $\alpha > 0$ and n large enough.

Theorem 1. *Let X be a binary shift. Then the entropy of X is positive if and only if X is independent over a set whose asymptotic density exists, is positive, and is equal to its Shnirelman density.*

This is a strengthening of [40, Theorem 8.1]. A general result applicable to all dynamical systems was proved by Glasner and Weiss [15] and Huang and Ye [17, Theorem 7.3]. Kerr and Li extended it further in [20]. We add the Shnirelman density to the picture, which shows that the independence set is even more structured. A similar result holds for more numerous alphabets (cf. [40, Theorem 8.3]). We state it without a proof.

Theorem 2. *Let $r \geq 2$ and X be a shift space over $\mathcal{A} = \{0, 1, \dots, r-1\}$. Then $h(X) > \log(r-1)$ if and only if X is independent over a set A whose asymptotic density exists, is positive, and is equal to its Shnirelman density.*

Before the proof we introduce all the necessary tools.

5.3. Limiting frequency. Our main technical tool is the limiting frequency.

Definition (Limiting frequency). Let \mathcal{A} be a finite alphabet and X be a shift space over \mathcal{A} . For every symbol $a \in \mathcal{A}$ and every point $x \in X$ we define the *characteristic set* $\chi_a(x)$ of a in x as the set of positions at which a appears in x , that is,

$$\chi_a(x) = \{j \in \mathbb{N} : x_j = a\}.$$

Let $w = w_1 \dots w_k$ and let $\|w\|_a$ denote the number of a 's in w , that is

$$\|w\|_a = |\{1 \leq j \leq k : w_j = a\}|.$$

Let $M_k^a(X)$ be the maximal number of occurrences of the symbol a among all blocks $w \in \mathcal{B}_k(X)$, that is,

$$M_k^a(X) = \max \{\|w\|_a : w \in \mathcal{B}_k(X)\}.$$

The sequence $\{M_k^a(X)\}_{k=1}^\infty$ is non-negative and subadditive, that is

$$0 \leq M_{m+n}^a(X) \leq M_m^a(X) + M_n^a(X)$$

for any positive integers m and n . By Fekete's Lemma the sequence $\{M_k^a(X)/k\}_{k=1}^\infty$ converges to its greatest lower bound. We call this limit the *limiting frequency of a in X* and denote it by

$$(2) \quad \text{Fr}_a(X) = \lim_{k \rightarrow \infty} \frac{M_k^a(X)}{k} = \inf_{k \geq 1} \frac{M_k^a(X)}{k}.$$

It can be shown that a limiting frequency of a symbol a is precisely the maximum of measures of the cylinder of a with respect to ergodic invariant measures supported on X . The next theorem is a strengthening of a well-known result about the ordinary density. It also follows² from the Ruzsa Theorem [33, Theorem 4, p. 323] (see also [30]).

Theorem 3. *Let X be a shift space over an alphabet \mathcal{A} . Then for every symbol $a \in \mathcal{A}$ there exists a point $\omega_a \in X$ such that*

$$d_{Sh}(\chi_a(\omega_a)) = d(\chi_a(\omega_a)) = \text{Fr}_a(X).$$

Proof. Clearly, $\text{Fr}_a(X) = 0$ implies $d_{Sh}(\chi_a(x)) = d(\chi_a(x)) = 0$ for all $x \in X$. We may therefore assume that $\text{Fr}_a(X) > 0$.

For every $m > 0$ let $\bar{w}^{(m)} = \bar{w}_1^{(m)} \dots \bar{w}_m^{(m)} \in \mathcal{B}_m(X)$ be a block which attains the maximal number of occurrences of the symbol a , that is,

$$\|\bar{w}^{(m)}\|_a = M_m^a(X) = \max\{\|w\|_a : w \in \mathcal{B}_m(X)\}.$$

We claim that for each integer $k > 0$ there exists a block $w^{(k)} \in \mathcal{B}_k(X)$ such that for all $1 \leq j \leq k$ we have

$$(3) \quad j \cdot \text{Fr}_a(X) \leq \|w_1^{(k)} \dots w_j^{(k)}\|_a.$$

In other words, we claim that given $k > 0$ we can find a word of length k such that for each $1 \leq j \leq k$ the average number of occurrences of a in the prefix $w_1^{(k)} \dots w_j^{(k)}$ of $w^{(k)}$ is not smaller than the limiting frequency of a .

For the proof of the claim, assume on the contrary that (3) does not hold for some $k > 0$. This means that every block w of length greater than or equal to k has a prefix of length $1 \leq j \leq k$ for which (3) fails, that is, such that

$$\frac{1}{j} \|w_1 \dots w_j\|_a < \text{Fr}_a(X) \leq 1.$$

Note that the left hand side of the above inequality is always a fraction of the form b/c , where $0 \leq b < c \leq k$. Since k is fixed, there are only finitely many such fractions, hence the number $\alpha_0 = \min\{\text{Fr}_a(X) - b/c > 0 : 0 \leq b < c \leq k\}$ is well-defined (the minimum exists) and is positive.

Take a positive integer N such that $\lfloor N\alpha_0 \rfloor = 1$. Since we assumed that our claim fails, and since $p = k^2N + 1$, we can chop the block $\bar{w}^{(p)}$ defined above into at least $Nk + 1$ pieces, each of length k at most, and for all but at most one of them (3) fails. To see it, we can imagine that we apply to the block $\bar{w}^{(p)}$ the following procedure: for the first step, we set $l(0) = 0$. Then we find the smallest j such that (3) fails for $\bar{w}^{(p)}$. By our assumption there is such j , moreover, $j \leq k$. We set $l(1) = j$ and define

$$\alpha_1 = \text{Fr}_a(X) - \frac{1}{j} \|\bar{w}_1^{(p)} \bar{w}_2^{(p)} \dots \bar{w}_j^{(p)}\|_a.$$

Clearly $\alpha_0 \leq \alpha_1$. For the next step we consider

$$v = \bar{w}_{j+1}^{(p)} \bar{w}_{j+2}^{(p)} \dots \bar{w}_p^{(p)},$$

that is, we chop off the prefix of length j from $\bar{w}^{(p)}$. We proceed inductively. Assume that we have performed s steps. This means that $l(0) < \dots < l(s)$ and

²We are indebted to Vitaly Bergelson for pointing this out.

$\alpha(1), \dots, \alpha(s)$ are defined for some $s \geq 1$. If $p - l(s) < k$, then we set $l(s+1) = p$ and we finish the construction. Otherwise, we have $p - l(s) \geq k$ and we consider

$$v = \bar{w}_{l(s)+1}^{(p)} \bar{w}_{l(s)+2}^{(p)} \dots \bar{w}_p^{(p)}.$$

Then we find the smallest j such that (3) fails for v . We set $l(s+1) = l(s) + j$ and

$$\alpha_{s+1} = \text{Fr}_a(X) - \frac{1}{l(s+1) - l(s)} \|\bar{w}_{[l(s)+1, l(s+1)]}^{(p)}\|_a \geq \alpha_0 > 0.$$

Let t denote the number of steps we have performed. We have found a strictly increasing sequence of integers $\{l(s)\}_{s=0}^{t+1}$ as well as a sequence of positive real numbers $\{\alpha_s\}_{s=1}^t$ such that $l(0) = 0$, $l(t+1) = p$, for every $j = 1, 2, \dots, t+1$ we have $0 \leq l(j) - l(j-1) \leq k$, and

$$(4) \quad \|\bar{w}_{[l(j-1)+1, l(j)]}^{(p)}\|_a = (l(j) - l(j-1))(\text{Fr}_a(X) - \alpha_j) \leq (l(j) - l(j-1))(\text{Fr}_a(X) - \alpha_0) \text{ for } j = 1, \dots, t.$$

Note that it may happen that $l(t) = l(t+1) = p$, but the inequality $\|\bar{w}_{[l(t)+1, p]}^{(p)}\|_a < k$ is valid regardless (if $l(t) + 1 > p$, then we agree to identify $\bar{w}_{[l(t)+1, p]}^{(p)}$ with the empty block). Clearly, we have $t \geq kN$, hence by the inequality in (4) and the definition of N we have

$$\begin{aligned} \|\bar{w}^{(p)}\|_a &= \left(\sum_{j=1}^t \|\bar{w}_{[l(j-1)+1, l(j)]}^{(p)}\|_a \right) + \|\bar{w}_{[l(t)+1, p]}^{(p)}\|_a \\ &\leq p(\text{Fr}_a(X) - \alpha_0) + k \leq p \text{Fr}_a(X) - k^2 N \alpha_0 + k < p \text{Fr}_a(X). \end{aligned}$$

Now we have the required contradiction, since $\|\bar{w}^{(p)}\|_a$ should be greater or equal to $p \text{Fr}_a(X)$.

We proved our claim is true, hence we have a sequence of blocks $\{w^{(k)}\}_{k=1}^\infty$, with $w^{(k)}$ from $\mathcal{B}_k(X)$, satisfying (3). We will use³ the $w^{(k)}$ to find a point $x \in X$ such that for each $k > 0$ the block $x_{[1, k]}$ is a prefix of some $w^{(l(k))}$ with $l(k) \geq k$.

For $k \in \mathbb{N}$ we inductively find a decreasing sequence S_k of subsets of \mathbb{N} so that for any $n \in S_k$ with $n \geq k$ all prefixes $w_{[1, k]}^{(n)}$ are equal. We begin by picking a symbol $a \in \mathcal{A}$ so that $w_1^{(k)} = a$ for infinitely many k 's. Let $S_1 = \{k \in \mathbb{N} : w_1^{(k)} = a\}$. Next, since the set of possible blocks of length 2 in X is finite, there is an infinite subset $S_2 \subset S_1$ such that the prefix $w_{[1, 2]}^{(n)}$ is the same for all $n \in S_2$. Continuing this way, we find for each $k \geq 2$ and infinite set $S_k \subset S_{k-1}$ so that all blocks $w_{[1, k]}^{(n)}$ are equal for $n \in S_k$. Define x to be the sequence with $x_k = w_k^{(n)}$ for some $n \in S_k$ with $n \geq k$. Note that then $x_{[1, k]} = w_{[1, k]}^{(n)}$ for every $n \in S_k \setminus \{1, \dots, k-1\}$ since $S_k \subset S_{k-1}$ for $k \geq 2$. Therefore for each $k > 0$ the block $x_{[1, k]}$ is a prefix of some $w^{(l(k))}$ with $l(k) \geq k$. In particular, $x_{[1, k]} \in \mathcal{B}(X)$ and hence $x \in X$.

For every $k \in \mathbb{N}$ we have

$$(5) \quad \text{Fr}_a(X) \leq \frac{1}{k} \|x_{[1, k]}\|_a \leq \frac{1}{k} M_k^a(X),$$

³This can be obtained easily by using a standard compactness argument, but the referee has encouraged us to present a completely elementary proof of this fact.

where the first inequality above is a consequence of the claim and the construction of the point x , while the second inequality follows from the definition of $M_k^a(X)$. Taking the infima over all $k \in \mathbb{N}$ of all terms in (5) and applying (2) we obtain

$$\text{Fr}_a(X) \leq \inf_{k \in \mathbb{N}} \frac{1}{k} \|x_{[1,k]}\|_a = d_{\text{Sh}}(\{j \in \mathbb{N} : x_j = a\}) \leq \inf_{k \in \mathbb{N}} \frac{1}{k} M_a^k(X) = \text{Fr}_a(X).$$

Since the values $M_a^k(X)/k$ converge to $\text{Fr}_a(X)$ as k goes to infinity (see (2)) we conclude that

$$\lim_{k \rightarrow \infty} \frac{1}{k} \|x_{[1,k]}\|_a = d(\{j \in \mathbb{N} : x_j = a\}) = d_{\text{Sh}}(\{j \in \mathbb{N} : x_j = a\}) = \text{Fr}_a(X). \quad \square$$

5.4. Some combinatorial Lemmas. The following lemma is well-known.

Lemma 1. *Let $0 < \varepsilon \leq 1/2$ and $n \geq 1$. Then*

$$\sum_{j=0}^{\lfloor n\varepsilon \rfloor} \binom{n}{j} \leq 2^{n \cdot H(\varepsilon)},$$

where $H(\varepsilon) = -\varepsilon \log \varepsilon - (1 - \varepsilon) \log(1 - \varepsilon)$.

Definition (Independence sets for blocks). Let \mathcal{F} be a (possibly empty) family of binary blocks of length $n \geq 0$. We say that \mathcal{F} is *independent over a set* $J \subset \mathbb{N}$ and J is an *independence set for* \mathcal{F} if for each map $\varphi : J \rightarrow \{0, 1\}$ there is a block $w \in \mathcal{F}$ whose i -th symbol is $\varphi(i)$ for every $i \in J$.

We denote the collection of all sets of independence for \mathcal{F} by $\mathcal{I}(\mathcal{F})$. We assume the convention that the empty set is a set of independence for every (including empty) family of n -blocks.

We will need a variant of the famous Sauer-Perles-Shelah Lemma (Lemma 3 below, see [34, 36]). The proof follows [2], where the next lemma (stated for sets) is attributed to Alain Pajor [29].

Lemma 2. *Let \mathcal{F} be a family of binary blocks of length $n \geq 0$. Then $|\mathcal{I}(\mathcal{F})| \geq |\mathcal{F}|$.*

Proof. The proof goes by induction on n . For $n = 0$ the situation is clear: $\mathcal{F} \subset \{\varepsilon\}$, hence $|\mathcal{F}| \leq 1$ and $|\mathcal{I}(\mathcal{F})| \geq 1$ because of the empty set.

Assume that the lemma holds for all families of n -blocks for some $n \geq 0$. Consider a family $\mathcal{F} \subset \{0, 1\}^{n+1}$. Let \mathcal{F}_0 be the family of prefixes of length n of all blocks in \mathcal{F} ending with 0. Analogously, we define \mathcal{F}_1 as the family of prefixes of length n of all blocks in \mathcal{F} ending with 1, that is,

$$\begin{aligned} \mathcal{F}_0 &= \{w \in \{0, 1\}^n : w0 \in \mathcal{F}\}, \\ \mathcal{F}_1 &= \{w \in \{0, 1\}^n : w1 \in \mathcal{F}\}. \end{aligned}$$

Since evidently, $|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1|$, we may apply the inductive assumption to infer that our lemma holds whenever $|\mathcal{F}_0| = 0$ or $|\mathcal{F}_1| = 0$. So assume that $|\mathcal{F}_0| \neq 0$ and $|\mathcal{F}_1| \neq 0$. Then every set which is independent for \mathcal{F}_0 or \mathcal{F}_1 is also independent for \mathcal{F} . Unfortunately, there are sets which are independent for both \mathcal{F}_0 and \mathcal{F}_1 (the empty set, for example). However, if $J \in \mathcal{I}(\mathcal{F}_0) \cap \mathcal{I}(\mathcal{F}_1)$, then $J \cup \{n+1\}$ is independent for \mathcal{F} , but $J \cup \{n+1\}$ is neither in $\mathcal{I}(\mathcal{F}_0)$ nor in $\mathcal{I}(\mathcal{F}_1)$. Therefore, by the inclusion-exclusion formula

$$|\mathcal{I}(\mathcal{F})| \geq |\mathcal{I}(\mathcal{F}_0) \cup \mathcal{I}(\mathcal{F}_1)| + |\mathcal{I}(\mathcal{F}_0) \cap \mathcal{I}(\mathcal{F}_1)| = |\mathcal{I}(\mathcal{F}_0)| + |\mathcal{I}(\mathcal{F}_1)|.$$

But applying the inductive hypothesis we obtain

$$|\mathcal{I}(\mathcal{F}_0)| + |\mathcal{I}(\mathcal{F}_1)| \geq |\mathcal{F}_0| + |\mathcal{F}_1| = |\mathcal{F}|$$

and the proof is finished. \square

Lemma 3. *Let $\mathcal{F} \subset \{0,1\}^n$ be a family of binary blocks of length $n \geq 1$. If for some $1 \leq k \leq n$ we have*

$$|\mathcal{F}| > \sum_{j=0}^{k-1} \binom{n}{j},$$

then \mathcal{F} is independent over some set of cardinality k .

Proof. By the previous lemma and our assumption we have

$$|\mathcal{I}(\mathcal{F})| \geq |\mathcal{F}| > \sum_{j=0}^{k-1} \binom{n}{j}.$$

As the last sum is the cardinality of the family of all subsets of $\{1, \dots, n\}$ with less than k elements there must be a set with k elements in $\mathcal{I}(\mathcal{F})$. \square

As a corollary of Sauer-Perles-Shelah Lemma and Lemma 1 we obtain a variant of the Karpovsky-Milman Lemma [19].

Lemma 4. *Let X be a binary shift with positive topological entropy. Then there is an $\varepsilon > 0$ such that for every $n \geq 1$ there is a set $J \subset \{1, \dots, n\}$ with $\lfloor \varepsilon n \rfloor$ elements which is an independence set for X .*

Proof. Take $0 < \varepsilon \leq 1/2$ such that $0 < H(\varepsilon) < h(X)$, where $H(\varepsilon) = -\varepsilon \log \varepsilon - (1 - \varepsilon) \log(1 - \varepsilon)$. By Lemma 1 we have

$$\sum_{j=0}^{\lfloor \varepsilon n \rfloor} \binom{n}{j} \leq 2^{n \cdot H(\varepsilon)} < 2^{nh(X)} \leq |\mathcal{B}_n(X)|,$$

for any $n \geq 1$. It follows from Lemma 3 that there is a set of independence for $\mathcal{B}_n(X)$ with $\lfloor \varepsilon n \rfloor$ elements. \square

5.5. Proof of Theorem 1.

Proof of Theorem 1. Let X be a binary shift.

Assume first that X has positive entropy. Let \mathcal{L}_n be the collection of characteristic functions of sets of independence for $\mathcal{B}_n(X)$. We can treat each element of \mathcal{L}_n as a binary block; then $\mathcal{L} = \bigcup_{n=1}^{\infty} \mathcal{L}_n$ is a factorial and prolongable binary language. We denote the shift space it defines by I_X .

By compactness of the full shift and the fact that a point $x \in \mathcal{A}^{\mathbb{N}}$ is in $X_{\mathcal{L}}$ if and only if $x_{[i,j]} \in \mathcal{L}$ for all $i, j \in \mathbb{N}$ with $i < j$, elements of I_X may be identified with characteristic functions of independence sets of X .

It follows from Lemma 4 that $\text{Fr}_1(I_X) > 0$. Using Theorem 3 we may now fix an element of I_X which is a characteristic function of J , an independence set for X , such that $d_{\text{Sh}}(J) = d(J) = \text{Fr}_1(I_X) > 0$.

If, on the other hand, J is an independence set for X such that $d_{\text{Sh}}(J) = d(J) > \delta > 0$, then for every $n \in \mathbb{N}$ there are at least $n\delta$ elements of J in $\{1, \dots, n\}$, and therefore $|\mathcal{B}_n(X)| \geq 2^{n\delta}$ for every $n \in \mathbb{N}$. This implies that $h(X) \geq \delta$ and completes the proof. \square

Remark. It follows from the above proof of Theorem 1 that for every binary shift X one has

$$h(X) \geq \text{Fr}_1(I_X).$$

One may wonder, whether the above inequality is, in fact, an equality. It holds, for example, for the square-free flow S (see [31]). Unfortunately, this is not always true. Consider the golden mean shift G (the shift defined by taking $\mathcal{F} = \{11\}$ as the set of forbidden blocks). Note that it is *hereditary*⁴, that is, given a word in its language, one can replace any number of 1's by 0's and the resulting word will still belong to the language. Therefore $I_G = G$. It is known that $h(G) = \log((1 + \sqrt{5})/2)$, but it is easy to see that $\text{Fr}_1(I_G) = 1/2$.

6. ENTROPY, TOPOLOGICAL MIXING, ERGODIC MEASURES, AND CHAOS

In this section we construct an example of a shift space with the same properties as the symbolic system constructed in [39] by Weiss. We begin by defining inductively a sequence of shift spaces X_k so that $X_0 \subset X_1 \subset \dots$ and these inclusions are strict. After completing the induction we define

$$X = \overline{\bigcup_{n=0}^{\infty} X_n}.$$

This means that $\mathcal{B}(X) = \mathcal{B}(X_0) \cup \mathcal{B}(X_1) \cup \mathcal{B}(X_2) \cup \dots$. It will be clear that at each step of our inductive construction some words are added to $\mathcal{B}(X_n)$ to form $\mathcal{B}(X_{n+1})$. We say that a block $u \in \{0, 1\}^*$ with $\|u\|_1 > 0$ is *added at step* $n \in \mathbb{N}$ if $u \in \mathcal{B}(X_n) \setminus \mathcal{B}(X_{n-1})$. Any block $u \in \mathcal{B}(X_0)$ with $\|u\|_1 > 0$ is *added at step* 0 by definition. Recall that $\|w\|_1$ denotes the number of 1's in a word w . A word w is a *from-1-to-1* word if $w = 1$, or $w = 1v1$ for some $v \in \{0, 1\}^*$. Let Y be a shift space over $\{0, 1\}$. Let $\mathcal{G}_n(Y)$ ($\mathcal{G}_n^*(Y)$) denote the set of all from-1-to-1 words in Y with at most (exactly) n occurrences of 1. The following two properties of blocks added at step n are easy consequences of the definition of X_n presented below. They will be proved along the lines of the definition.

Lemma 5. *If a block u is added at step $n \geq 1$, then there is a prefix of u of the form $0^\alpha \bar{u}' 0^\beta \bar{u}''$ for some $\bar{u}', \bar{u}'' \in \mathcal{G}_{2^{n-1}}(X_{n-1})$, $\alpha \geq 0$ and $\beta \geq 2^{n-2}$. It follows that:*

- (1) $2^{n-1} < \|u\|_1$;
- (2) $\frac{\|u\|_1}{|u|} \leq \frac{2^n}{2^n + 2^{2n-2}} = \frac{1}{1 + 2^{n-2}}.$

6.1. Construction of X . We inductively construct an increasing sequence of shift spaces $X_0 \subset X_1 \subset \dots$ and then define X as the closure of the union of all X_n 's.

Outline of the construction. Recall that a shift space X is mixing if for every blocks $u, v \in \mathcal{B}(X)$ there is an $N \in \mathbb{N}$ such that for each $n \geq N$ one can find a block w of length n such that $uwv \in \mathcal{B}(X)$. We call such a word w the *n -transition block* from u to v . The periodic points are dense in a shift space X is for every block $u \in \mathcal{B}(X)$ there is a block v such that $(uv)^\infty \in X$. If $x \in X$ is a periodic point, then the *prime period* of x is the length of the shortest block $w \in \mathcal{B}(X)$ satisfying $x = w^\infty$.

⁴Hereditary shifts were introduced in [20] and examined in [22].

In our construction the n -transition block can be always a block of 0's. The length N depends on the number of 1's which occur in u and v , namely we choose N such that the relative density of 1's in $u0^N v$ is small. The periodic points are dense in our shift space but the density of 1's in a periodic point decreases very fast when the prime period of the point grows. We achieve this by adding periodic points of the form $x = (u0^n)^\infty$ with n large enough to force the relative frequency of 1's in x to be small.

Definition of X_0 . We define X_0 to be the set of all sequences in the full shift over $\{0, 1\}$ with at most one appearance of the symbol 1. That is, $X_0 = \{0^\infty\} \cup \{0^\alpha 10^\infty : \alpha \geq 0\}$. Note that

$$\mathcal{B}(X_0) = \{0^\alpha : \alpha \geq 0\} \cup \{0^\alpha 10^\beta : \alpha, \beta \geq 0\}.$$

Before we present the details in full generality let us first work out the special case of X_1 .

Definition of X_1 . We add to X_0 the orbits of points of the form:

$$(10^\beta)^\infty, \quad \text{or} \quad 0^\alpha 10^\beta 10^\infty, \quad \text{where } \beta \geq 1, \alpha \geq 0.$$

Note that every block u added at step 1 has at least two occurrences of the symbol 1, and fulfills

$$\|u\|_1/|u| \leq \frac{2}{3}.$$

The bound above is the best possible, since it is attained by the block 101 added at step 1. Therefore all assertions of Lemma 5 hold.

Definition of X_{n+1} . For the inductive step, given X_n and $n \geq 0$, we construct a shift space Y_{n+1} and set $X_{n+1} = X_n \cup Y_{n+1}$. To define Y_{n+1} we specify a set of auxiliary points and then let Y_{n+1} be the closure of the set of auxiliary points.

There will be two types of auxiliary points. We call them *periodic* and *joining* auxiliary points. Let P'_{n+1} be the set of all periodic points of the form $(u0^k)^\infty$, where $k \geq 2^{2n}$ and $u \in \mathcal{G}_{2^n}(X_n)$. A point x is a periodic auxiliary point if it belongs to the orbit of some point in P'_{n+1} . We denote the set of auxiliary periodic points by P_{n+1} . We have

$$P_{n+1} = \bigcup_{l=0}^{\infty} \sigma^l(\{(u0^k)^\infty : k \geq 2^{2n}, u \in \mathcal{G}_{2^n}(X_n)\}).$$

It is clear that P_{n+1} is shift invariant. The set of joining auxiliary points is given by $J_{n+1} = J'_{n+1} \cup J''_{n+1}$, where

$$\begin{aligned} J'_{n+1} &= \{0^\alpha u 0^\infty : \alpha \geq 0, u \in \mathcal{G}_{2^n}(X_n)\}, \\ J''_{n+1} &= \{0^\alpha u 0^\beta v 0^\infty : \alpha \geq 0, \beta \geq 2^{2n}, u, v \in \mathcal{G}_{2^n}(X_n)\}. \end{aligned}$$

Points from J''_{n+1} guarantee that the mixing condition holds for pairs of blocks u, v with at most 2^n occurrences of 1. But J''_{n+1} is not shift invariant and orbits of points from J''_{n+1} end up in $J'_{n+1} \cup \{0^\infty\}$ after a finite number of shift operations.

We claim that

$$\begin{aligned} (6) \quad \overline{P}_{n+1} &\subset P_{n+1} \cup \overline{J}_{n+1}, \\ (7) \quad \overline{J}_{n+1} &= J_{n+1} \cup \{0^\infty\}. \end{aligned}$$

For the proof of the inclusion in (6) assume that a sequence $\{x^{(m)}\}_{m=1}^\infty \subset P_{n+1}$ converges to x . Without loss of generality we can assume that

$$x^{(m)} = u_{[t(m), \ell(m)]}^{(m)} \left(u^{(m)} 0^{\beta_m} \right)^\infty$$

where $u^{(m)} \in \mathcal{G}_{2^n}(X_n)$ is a block of length $\ell(m)$, $1 \leq t(m) \leq \ell(m)$ and $\beta_m \geq 2^{2m}$. We also assume that $u^{(m)} 0^{\beta_m}$ is primitive, that is, it is not a concatenation of two or more copies of some block over $\{0, 1\}$.

There are two possibilities:

Case 1: $\mathcal{G}_{2^{n+1}}^*(x) \neq \emptyset$. Let N be the smallest integer such that $x_{[1, N]} \in \mathcal{G}_{2^{n+1}}^*(x)$. But $x^{(m)} \rightarrow x$ as $m \rightarrow \infty$, thus for all sufficiently large m we have $x_{[1, N]}^{(m)} = x_{[1, N]}$. In particular, $x_{[1, N]} \in \mathcal{B}(x^{(m)})$. Consequently, $u^{(m)} 0^{\beta_m}$ is a subblock of $x_{[1, N]}$. There are only finitely many subblock of $x_{[1, N]}$ of the form $u 0^k$ where $k \geq 2^m$ and $u \in \mathcal{G}_{2^n}(X_n)$. Therefore there are only finitely many $y \in P_{n+1}$ such that $\mathcal{G}_{2^{n+1}}^*(y)$ contains $x_{[1, N]}$. It follows that $x^{(m)}$ is eventually constant, hence $x \in P_{n+1}$.

Case 2: $\mathcal{G}_{2^{n+1}}^*(x) = \emptyset$. It is easy to see that in this case $x \in \overline{J}_{n+1}$.

The proof of (7) follows the same lines as the proof above. We leave the details to the reader.

We set

$$Y_{n+1} = \overline{P_{n+1} \cup J_{n+1}} \text{ and } X_{n+1} = X_n \cup Y_{n+1}.$$

Clearly, $X_n \subset X_{n+1}$. This completes the induction step. Observe that any block u added at step $n+1$ has a prefix $0^\alpha \bar{u}' 0^\beta \bar{u}''$ for some $\bar{u}', \bar{u}'' \in \mathcal{G}_{2^n}(X_n)$, $\alpha \geq 0$, $\beta \geq 2^{2n}$. Furthermore, u fulfills $\|u\|_1 > 2^n$ as otherwise u would be added at earlier step. Therefore

$$\frac{\|u\|_1}{|u|} \leq \frac{\|\bar{u}'\|_1 + \|\bar{u}''\|_1}{|\bar{u}'| + |\bar{u}''| + \beta}.$$

But $\|\bar{u}'\|_1$ and $\|\bar{u}''\|_1$ are bounded by 2^n , and we conclude that

$$\frac{\|u\|_1}{|u|} \leq \frac{2^{n+1}}{2^{n+1} + 2^{2n}} = \frac{1}{1 + 2^{n-1}},$$

as claimed in Lemma 5. We note the following consequence of the formulas (7) and (6).

Corollary 1. *If $x \in X_n$ and the set $\{j \in \mathbb{N} : x_j = 1\}$ is infinite, then x must be a periodic point.*

6.2. Properties of X . We describe the properties of X as a sequence of lemmas.

Lemma 6. *Periodic points are dense in X .*

Proof. Take any $u \in \mathcal{B}(X)$. Assume that u is added at step n and let $k = \|u\|_1$. If $n = k = 0$, then 0^∞ is a periodic point in the cylinder of u . If $k > 0$, then we write $u = 0^\alpha \bar{u} 0^\beta$ for some $\bar{u} \in \mathcal{G}_k(X)$, $\alpha, \beta \geq 0$. We set $m = \lceil \log_2 k \rceil$ and note that $m \geq n$. Hence $\bar{u} \in \mathcal{G}_k(X_n) \subset \mathcal{G}_{2^m}(X_m)$. We may infer that

$$0^\alpha \left(\bar{u} 0^{\alpha+\beta+2^{2m}} \right)^\infty = (u 0^{2^{2m}})^\infty \in P_{m+1},$$

and therefore there is a periodic point of X in the cylinder of u . \square

Lemma 7. *The shift space X is topologically mixing.*

Proof. Let u and v be two nonempty blocks in $\mathcal{B}(X)$. Assume that $\|u\|_1 > 0$ and $\|v\|_1 > 0$. Consequently, there is a $k \geq 0$ such that $u = 0^a \bar{u} 0^b$ and $v = 0^c \bar{v} 0^d$ for some $a, b, c, d \geq 0$ and $\bar{u}, \bar{v} \in \mathcal{G}_{2^k}(X_k)$. Therefore $u 0^\beta v 0^\infty \in J_{n+1} \subset X$ for all $\beta \geq 2^{2^k}$. If for some $\alpha \in \mathbb{N}$ we have $u = 0^\alpha$ or $v = 0^\alpha$, then our reasoning is similar. Therefore X is topologically mixing. \square

Lemma 8. *If $x \in X$, then either x is periodic or the asymptotic density of the set $\chi_1(x) = \{j \in \mathbb{N} : x_j = 1\}$ equals zero.*

Proof. Assume that the set $\chi_1(x)$ is infinite. Denote by λ_n the step at which $x_{[1,n]}$ was added. The sequence λ_n is nondecreasing, hence it is either bounded, or $\lambda_n \nearrow \infty$. In the former case $x \in X_m$ for some $m \in \mathbb{N}$, hence x must be periodic. In the later case we have

$$\frac{|\{1 \leq j \leq n : x_j = 1\}|}{n} \leq \frac{1}{1 + 2^{\lambda_n - 2}},$$

which monotonically tends to zero as n goes to infinity by Lemma 5. \square

Lemma 9. *Every point $x \in X$ is a generic point for some σ -invariant ergodic measure μ_x on X and all ergodic invariant measures of (X, σ) are supported on single periodic points.*

Proof. If x is a periodic point, then it is clear that x is generic. If x is not a periodic point, then for each $k \in \mathbb{N}$ the asymptotic density of the set $\chi_1^k(x) = \{j \in \mathbb{N} : x_{[j, j+k)} = 0^k\}$ equals one. It follows that x is generic for the Dirac measure concentrated on the fixed point 0^∞ . It follows that there are no other ergodic invariant measures. \square

Lemma 10. *The shift space X has no DC3 pair.*

Proof. Fix $k \in \mathbb{N}$ and take any $x, y \in X$. We define

$$\text{Eq}^k(x, y) = \{j \in \mathbb{N} : x_{[j, j+k)} = y_{[j, j+k)}\}.$$

It is easy to see that in the case of symbolic dynamical systems equipped in the metric ρ defined by (1) for each $k \in \mathbb{N}$ and $t \in (2^{-k}, 2^{-k+1}]$ we have

$$(8) \quad F_{xy}(t) = \underline{d}(\text{Eq}^k(x, y)) \quad \text{and} \quad F_{xy}^*(t) = \bar{d}(\text{Eq}^k(x, y)).$$

We will show that $\text{Eq}^k(x, y)$ has a well defined asymptotic density. Let $u_j = x_{[j, j+k)}$ and $v_j = y_{[j, j+k)}$ for $j \in \mathbb{N}$. If x and y are periodic, then obviously $\text{Eq}^k(x, y)$ has an asymptotic density for each k . For the remaining case we assume, without loss of generality, that x is generic for the Dirac mass at 0^∞ . Observe that

$$\begin{aligned} \text{Eq}^k(x, y) &= \{j \in \mathbb{N} : u_j = v_j\} \\ &= \{j \in \mathbb{N} : u_j = 0^k = v_j\} \cup \{j \in \mathbb{N} : u_j = v_j \neq 0^k\}. \end{aligned}$$

The set $\{j \in \mathbb{N} : u_j = v_j \neq 0^k\}$ is contained in $\{j \in \mathbb{N} : u_j \neq 0^k\}$, hence has asymptotic density zero. We also know that $d(\{j \in \mathbb{N} : u_j = 0^k\}) = 1$ and $d(\{j \in \mathbb{N} : v_j = 0^k\})$ exists, therefore $\text{Eq}^k(x, y)$ has a well defined asymptotic density. Since k was arbitrary, we apply (8) to conclude that $F_{xy}(t) = F_{xy}^*(t)$ for each $t \in \mathbb{R}$. \square

The following result may be proved directly, or using a result of Downarowicz [10], who showed that any positive entropy system has many DC2-pairs.

Lemma 11. *The shift space X has topological entropy zero.*

Proof. Note that all ergodic measures of X are concentrated on periodic points, hence their measure-theoretic entropy is equal zero. By the variational principle the topological entropy is also equal zero. \square

Finally, we note the following corollary of Lemma 9.

Corollary 2. *There is no ergodic invariant measure with support X .*

Lemma 12. *The shift space X is not ω^* -chaotic.*

Proof. The only minimal sets in X are periodic orbits. Therefore no ω -limit set contains an infinite minimal set. \square

The following lemma is well-known (see [21, Chap. 4, Problem 41]).

Lemma 13. *There exists an uncountable set Γ and a family $\{B(\gamma)\}_{\gamma \in \Gamma}$ of infinite subsets of \mathbb{N} such that for any $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$ we have*

- (1) $B(\alpha) \cap B(\beta)$ is infinite,
- (2) $B(\alpha) \setminus B(\beta)$ is infinite.

We now proceed to our last result.

Lemma 14. *The shift space X is ω -chaotic.*

Proof. We first define for any infinite set $S \subset \mathbb{N}$ an infinite transitive subsystem $X_S \subset X$. To do this we proceed as with the construction of X . We start with $X_0^S = X_0$. During the induction step, we define the set of joining auxiliary points by $J_{n+1}^S = J_{n+1}^S \cup \check{J}_{n+1}^S$, where

$$\begin{aligned} j_{n+1}^S &= \{0^\alpha u 0^\infty : \alpha \geq 0, u \in \mathcal{G}_{2^n}(X_n^S)\}, \\ \check{j}_{n+1}^S &= \{0^\alpha u 0^\beta v 0^\infty : \alpha \geq 0, \beta \geq 2^{2^n}, \beta \in S, u, v \in \mathcal{G}_{2^n}(X_n^S)\}. \end{aligned}$$

Note that $J_{n+1}^S \subset J_{n+1}$ and J_{n+1}^S nonempty because S is infinite. The main difference from the previous construction is the condition $\beta \in S$. We do not add any periodic points. As above we set $X_{n+1}^S = X_n^S \cup \overline{J_{n+1}^S}$. Then

$$X_S = \bigcup_{n=0}^{\infty} X_n^S.$$

We inductively define the family of *allowed* blocks. We say that every block $w \in \{0, 1\}^*$ with $\|w\|_1 \leq 1$ is allowed. Assume that for some $n \in \mathbb{N}$ we have defined which blocks w over $\{0, 1\}$ with at most 2^{n-1} ones are allowed. We say that a block $w \in \{0, 1\}^*$ with $2^{n-1} < \|w\|_1 \leq 2^n$ is allowed if there are allowed from-1-to-1 blocks u_1 and u_2 with $1 \leq \|u_j\|_1 \leq 2^{n-1}$ for $j = 1, 2$, $a, b \geq 0$, and $\beta \in S$ such that $\beta \geq 2^{2^n}$ and $w = 0^a u_1 0^\beta u_2 0^b$. A straightforward (but tedious) analysis of the definition of X_S shows that block w belongs to the language $\mathcal{B}(X_S)$ of X_S if and only if it is an allowed block. This finishes the construction of X_S .

We claim that X_S is transitive. To this end take two allowed blocks u and v . Without loss of generality we may assume that u and v are from-1-to-1 blocks. Let $i = \|u\|_1$ and $j = \|v\|_1$. Pick N such that $\max\{i, j\} \leq 2^N$. Let $\beta \in S$ be such that $\beta \geq 2^{2^N}$. Then $u 0^\beta v$ is an allowed word, which proves the transitivity of X_S . Again, we used the fact that S is infinite. It follows that X_S is a topologically transitive and infinite shift space. Therefore X_S is uncountable.

Note that if S and T are infinite subsets of \mathbb{N} such that $S \cap T$ is also infinite, then $X_{S \cap T} \subset X_S \cap X_T$ (here one can write an equality, but this inclusion suffices for our purposes). It is also clear that if $S \subset T \subset \mathbb{N}$ is infinite, then $X_S \subset X_T$. Furthermore, if $S, T \subset \mathbb{N}$ are infinite and disjoint, then $X_S \cap X_T = X_0$. To see this, observe that X_0 is contained in X_S for every infinite S . On the other hand, if $x \in (X_S \cap X_T) \setminus X_0$, then we can find $w = 10^\beta 1 \in \mathcal{B}(X_S) \cap \mathcal{B}(X_T)$, which is impossible since then $\beta \in S \cap T = \emptyset$. The same argument shows that if $S \neq T$, then $X_S \neq X_T$.

Recall that X_0 is a countable shift space. In particular, if $S, T \subset \mathbb{N}$ are infinite and disjoint, then $X_S \setminus X_T = X_S \setminus X_0$ is uncountable.

Let $\{B(\gamma)\}_{\gamma \in \Gamma}$ be a family provided by Lemma 13. For any $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$ we define $C(\alpha, \beta) = B(\alpha) \cap B(\beta)$ and $D(\alpha, \beta) = B(\alpha) \setminus B(\beta)$.

Let X_γ be the shift space X_S constructed for $S = B(\gamma)$, where $\gamma \in \Gamma$. Using transitivity, for any $\gamma \in \Gamma$ we pick a point $x_\gamma \in X_\gamma$ such that its omega-limit set, $\omega(x_\gamma) = X_\gamma$. We claim that $\{x_\gamma\}_{\gamma \in \Gamma}$ is an ω -chaotic set. Clearly, $\{x_\gamma\}_{\gamma \in \Gamma}$ is uncountable, because $X_\gamma \neq X_{\gamma'}$ if $\gamma \neq \gamma'$. Furthermore, $B(\gamma)$ is infinite, therefore $\omega(x_\gamma)$ is uncountable for each $\gamma \in \Gamma$ and is not contained in the set of periodic points. Now note that for every $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$ we have $\omega(x_\alpha) \cap \omega(x_\beta) \supset X_{C(\alpha, \beta)}$, which is uncountable because $C(\alpha, \beta)$ is infinite and $\omega(x_\alpha) \setminus \omega(x_\beta)$ is also uncountable (it contains all transitive points $X_{D(\alpha, \beta)}$, because $D(\alpha, \beta)$ is infinite and disjoint with $B(\beta)$). Hence $\{x_\gamma : \gamma \in \Gamma\}$ is an ω -chaotic set, which proves that X is ω -chaotic. \square

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REFERENCES

- [1] Adler, R. L.; Konheim, A. G.; McAndrew, M. H. Topological entropy. *Trans. Amer. Math. Soc.* 114 1965 309–319.
- [2] Anstee, R. P.; Rónyai, Lajos; Sali, Attila. Shattering news. *Graphs Combin.* 18 (2002), no. 1, 59–73.
- [3] F. Balibrea, J. Smítal and M. Štefánková, *The three versions of distributional chaos*, *Chaos Solitons Fractals*, **23** (2005), 1581–1583.
- [4] Banks, J.; Brooks, J.; Cairns, G.; Davis, G.; Stacey, P. On Devaney’s definition of chaos. *Amer. Math. Monthly* 99 (1992), no. 4, 332–334.
- [5] Blanchard, François. Topological chaos: what may this mean? *J. Difference Equ. Appl.* 15 (2009), no. 1, 23–46.
- [6] Blanchard, François; Glasner, Eli; Kolyada, Sergii; Maass, Alejandro. On Li-Yorke pairs. *J. Reine Angew. Math.* 547 (2002), 51–68.
- [7] Blanchard, François; Huang, Wen. Entropy sets, weakly mixing sets and entropy capacity. *Discrete Contin. Dyn. Syst.* 20 (2008), no. 2, 275–311.

- [8] Blanchard, François; Huang, Wen; Snoha, L'ubomír. Topological size of scrambled sets. *Colloq. Math.* 110 (2008), no. 2, 293–361.
- [9] Devaney, Robert L. An introduction to chaotic dynamical systems. Second edition. Addison-Wesley Studies in Nonlinearity. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989.
- [10] Downarowicz, T. Positive topological entropy implies chaos DC2. *Proc. Amer. Math. Soc.* 142 (2014), no. 1, 137–149.
- [11] Downarowicz, Tomasz; Ye, Xiangdong. When every point is either transitive or periodic. *Colloq. Math.* 93 (2002), no. 1, 137–150.
- [12] Furstenberg, Harry. Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. *Math. Systems Theory* 1 1967 1–49.
- [13] Glasner, Eli; Ye, Xiangdong. Local entropy theory. *Ergodic Theory Dynam. Systems* 29 (2009), no. 2, 321–356.
- [14] Glasner, Eli; Weiss, Benjamin. Sensitive dependence on initial conditions. *Nonlinearity* 6 (1993), no. 6, 1067–1075.
- [15] Glasner, Eli; Weiss, Benjamin. Quasi-factors of zero-entropy systems. *J. Amer. Math. Soc.* 8 (1995), no. 3, 665–686.
- [16] Huang, Wen; Ye, Xiangdong. Devaney's chaos or 2-scattering implies Li-Yorke's chaos. *Topology Appl.* 117 (2002), no. 3, 259–272.
- [17] Huang, Wen; Ye, Xiangdong. A local variational relation and applications. *Israel J. Math.* 151 (2006), 237–279.
- [18] Huang, Wen; Li, Jian; Ye, Xiangdong. Stable sets and mean Li-Yorke chaos in positive entropy systems. *J. Funct. Anal.* 266 (2014), no. 6, 3377–3394.
- [19] Karpovsky, M. G.; Milman, V. D. Coordinate density of sets of vectors. *Discrete Math.* 24 (1978), no. 2, 177–184.
- [20] Kerr, David; Li, Hanfeng. Independence in topological and C^* -dynamics. *Math. Ann.* 338 (2007), no. 4, 869–926.
- [21] Komjáth, Péter; Totik, Vilmos. Problems and theorems in classical set theory. Problem Books in Mathematics. Springer, New York, 2006. xii+514 pp.
- [22] Kwietniak, Dominik. Topological entropy and distributional chaos in hereditary shifts with applications to spacing shifts and beta shifts. *Discrete Contin. Dyn. Syst.* 33 (2013), no. 6, 2451–2467.
- [23] Li, Jian; Ye, Xiangdong. *Recent development of chaos theory in topological dynamics*, to appear in *Acta Math. Sin. (Engl. Ser.)*
- [24] Li, Shi Hai. ω -chaos and topological entropy. *Trans. Amer. Math. Soc.* **339** (1993), no. 1, 243–249.
- [25] Li, Shi Hai. *Dynamical properties of the shift maps on the inverse limit spaces*. *Ergodic Theory Dynam. Systems* **12** (1992), no. 1, 95–108.
- [26] Li, Tien Yien; Yorke, James A. Period three implies chaos. *Amer. Math. Monthly* 82 (1975), no. 10, 985–992.
- [27] Lind, Douglas; Marcus, Brian. An introduction to symbolic dynamics and coding. *Cambridge University Press, Cambridge*, 1995.
- [28] Oprocha, Piotr. Relations between distributional and Devaney chaos. *Chaos* 16 (2006), no. 3, 033112, 5 pp.
- [29] Pajor, Alain. Sous-espaces l_1^n des espaces de Banach. (French) [l_1^n -subspaces of Banach spaces] With an introduction by Gilles Pisier. Travaux en Cours [Works in Progress], 16. Hermann, Paris, 1985.
- [30] Peres, Yuval. A combinatorial application of the maximal ergodic theorem. *Bull. London Math. Soc.* 20 (1988), no. 3, 248–252.
- [31] Peckner, Ryan. *Uniqueness of the measure of maximal entropy for the squarefree flow*. Preprint (2014) [arXiv:1205.2905v6](https://arxiv.org/abs/1205.2905v6)[math.DS]
- [32] Piśkuła, Rafał. On enveloping semigroups of almost one-to-one extensions of minimal group rotations. *Colloq. Math.* 129 (2012), no. 2, 249–262.
- [33] Ruzsa, I. Z. On difference sets. *Studia Sci. Math. Hungar.* 13 (1978), no. 3-4, 319–326 (1981).
- [34] Sauer, N. On the density of families of sets. *J. Combinatorial Theory Ser. A* 13 (1972), 145–147.
- [35] Schweizer, B.; Smítal, J. Measures of chaos and a spectral decomposition of dynamical systems on the interval. *Trans. Amer. Math. Soc.* 344 (1994), no. 2, 737–754.

- [36] Shelah, Saharon. A combinatorial problem; stability and order for models and theories in infinitary languages. *Pacific J. Math.* 41 (1972), 247–261.
- [37] Smítal, J. Chaotic functions with zero topological entropy. *Trans. Amer. Math. Soc.* 297 (1986), no. 1, 269–282.
- [38] Walters, Peter. An introduction to ergodic theory. *Graduate Texts in Mathematics*, 79. Springer-Verlag, New York-Berlin, 1982.
- [39] Weiss, Benjamin. Topological transitivity and ergodic measures. *Theory of Computing Systems* 5.1 (1971): 71–75.
- [40] Weiss, Benjamin. Single orbit dynamics. *CBMS Regional Conference Series in Mathematics*, 95. American Mathematical Society, Providence, RI, 2000.
- [41] Xiong, Jin Cheng. A chaotic map with topological entropy. *Acta Math. Sci. (English Ed.)* 6 (1986), no. 4, 439–443.

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